A class of inequalities inducing new separability criteria for bipartite quantum systems

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Abstract. Inspired by the realignment or computable cross norm criterion, we present a new result about the characterization of quantum entanglement. Precisely, an interesting class of inequalities satisfied by all separable states of a bipartite quantum system is derived. These inequalities induce new separability criteria that generalize the realignment criterion.

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1. Introduction

Entanglement is a very peculiar and essential feature of quantum theory, as recognized since the early stages of development of the theory by Einstein, Podolsky and Rosen [1], and by Schrödinger [2, 3] (who introduced the german term 'Verschränkung' and translated it into English as 'entanglement'). Recently, entanglement has been investigated with a renewed interest motivated by ideas and applications stemming from the field of quantum information science [4]. Indeed, nowadays quantum entanglement is not only regarded as a fundamental key for the interpretation of quantum mechanics but also as an important resource for quantum information, communication and computation tasks [5]. However, despite the great efforts made by the scientific community in the past decades, there are still several open issues regarding the mathematical characterization of entangled quantum states: the study of *bipartite* entanglement is already a difficult, extensive and rapidly evolving subject, even in the case of quantum systems with a finite number of levels, and *multipartite* entanglement is still a rather obscure matter (see, for instance, the review papers [6, 7] and references therein). In the present contribution, our discussion will be restricted to the case of bipartite systems with a finite number of levels.

According to the definition due to Werner [8], *entangled* (mixed) states differ from *separable* states since they cannot be prepared using only local operations and classical communication; hence, they may exhibit non-classical correlations. In mathematical terms, a mixed state $\hat{\rho}$ — a positive (trace class) operator of unit trace — in a composite Hilbert space

$\mathcal{H} = \mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$ is called *separable* if it can be represented as a convex sum of product states (the sum converging with respect to the trace norm):

$$\hat{\rho} = \sum_i p_i \; \hat{\rho}_i^{\mathsf{A}} \otimes \hat{\rho}_i^{\mathsf{B}}, \tag{1}$$

with $p_i > 0$ and $\sum_i p_i = 1$; otherwise, $\hat{\rho}$ is said to be *nonseparable* or *entangled*. We remark that, if $\hat{\rho}$ is a separable state, decomposition (1) — a *separability decomposition* of the state $\hat{\rho}$ — is in general not unique, and it can be assumed to be a *finite* sum if the Hilbert space $\mathcal{H}$ is finite-dimensional. The smallest number of terms in the sum (usually called *cardinality* of the separable state $\hat{\rho}$) is not larger than the squared dimension of the *total* Hilbert space of the system $\mathcal{H}$ (see [9]).

Since quantum entanglement, as already mentioned, is a very important subject (also in view of its potential applications), separability criteria are extremely precious tools. As far as we know, separability criteria found so far fall into two classes: on one hand, criteria based on both necessary and sufficient conditions, but not practically implementable; on the other hand, criteria that are relatively easy to apply, but rely on only necessary, or only sufficient, conditions. Among necessary and sufficient criteria, we mention the 'positive maps criterion' [10], which leads to the 'Peres-Horodecki criterion' [10, 11] for 2×2 and 2×3 bipartite systems (in these special cases, we have an operational necessary and sufficient criterion), and the 'contraction criterion' [12]. Among necessary conditions for separability (or, equivalently, sufficient conditions for entanglement), it is worth mentioning the celebrated 'PPT criterion' [11] (which, in the special cases of 2×2 and 2×3 bipartite systems, is precisely the Peres-Horodecki necessary-sufficient criterion), the 'reduction criterion' [13, 14], the 'majorization criterion' [15], and the criterion that was proposed in ref. [16] with the name of 'realignment criterion' (RC) and in ref. [17] with the name of 'computable cross norm criterion', which will be central in the present contribution.

In this paper, we reconsider the RC and, using essentially the same tools that allow to prove this criterion, we obtain a new class of inequalities satisfied by all separable states of a bipartite quantum system. These inequalities potentially induce new separability criteria; i.e. a certain inequality produces a separability criterion if it makes sense to check it for a generic state and it is actually violated by some state. Every state that violates such an inequality is then nonseparable. Numerical calculations show that from the class of inequalities introduced in the present paper one obtains, indeed, a wide class of new separability criteria. This class contains, in particular, the original RC and a powerful separability criterion which is the main result obtained in ref. [18]. Other remarkable particular cases are given by the 'enhancement by local filtering operations' of the RC and of the criterion of ref. [18], see Sect. 5. All the new criteria share with the RC the important property of being easily implementable, and they are, in general, independent of the original RC. For instance, it can be shown that the criterion derived in ref. [18] is stronger than the standard RC.

The paper is organized as follows. In Sect. 2, the RC is reviewed starting from the Schmidt decomposition of pure states. In Sect. 3, by an argument similar to the one that allows to prove the RC, we derive the announced class of inequalities. As mentioned before, this class of inequalities induce new separability criteria; a few simple examples of entanglement

detection are given in Sect. 4. Finally, in Sect. 5, conclusions are drawn.

2. From the Schmidt decomposition to the realignment criterion

Aim of the present section is to review some known facts about the *realignment criterion* (RC) for bipartite quantum systems. We will try, in particular, to highlight the relation between the *Schmidt decomposition* [19] of a bipartite (pure or mixed) state and the RC. This relation will be our starting point for establishing novel separability criteria related to the RC.

Let us consider a bipartite quantum system with carrier Hilbert space $\mathcal{H} = \mathcal{H}_\mathsf{A} \otimes \mathcal{H}_\mathsf{B}$, with $\mathcal{H}_\mathsf{A} \cong \mathbb{C}^{N_\mathsf{A}}$ and $\mathcal{H}_\mathsf{B} \cong \mathbb{C}^{N_\mathsf{B}}$, where we assume that $2 \leq N_\mathsf{A}, N_\mathsf{B} < \infty$. We will fix in the 'local Hilbert spaces' $\mathcal{H}_\mathsf{A}, \mathcal{H}_\mathsf{B}$ orthonormal bases $\{|n\rangle\}_{n=1,\ldots,N_\mathsf{A}}$ and $\{|\nu\rangle\}_{\nu=1,\ldots,N_\mathsf{B}}$, respectively (notice that we use Latin indexes for the subsystem A and Greek indexes for the subsystem B). Then, a state vector $|\psi\rangle \in \mathcal{H}$ of the composite system can be written as $|\psi\rangle = \sum_{n,\nu} \psi_{n\nu} |n\rangle|\nu\rangle$ (for notational conciseness, we will occasionally omit the tensor product symbol). It is clear that one can regard the components of the vector $|\psi\rangle$ in the given basis as the entries of a $N_\mathsf{A} \times N_\mathsf{B}$ matrix ψ:

$$\psi \equiv [\psi_{n\nu}]\,; \tag{2}$$

here notice that the Latin and the Greek indexes play respectively the role of row and column indexes of the matrix ψ.

For instance, in the simplest case of a two-qubit system — i.e. $\mathcal{H}_\mathsf{A} \cong \mathcal{H}_\mathsf{B} \cong \mathbb{C}^2$ — with a state vector

$$|\psi\rangle = \psi_{11}|11\rangle + \psi_{12}|12\rangle + \psi_{21}|21\rangle + \psi_{22}|22\rangle \in \mathcal{H}_\mathsf{A} \otimes \mathcal{H}_\mathsf{B} \tag{3}$$

is associated a 2×2 the matrix

$$\psi = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}. \tag{4}$$

As it is well known, a Schmidt decomposition (non-uniquely determined) of the state vector $|\psi\rangle$ comes from a singular value decomposition (SVD) [20] of the matrix of coefficients (2):

$$\psi = U\,\Delta\,V = \left[\psi_{n\nu} = \sum_{m,\mu} U_{nm}\,\Delta_{m\mu} V_{\mu\nu}\right] \tag{5}$$

where U and V are respectively $N_\mathsf{A} \times N_\mathsf{A}$ and $N_\mathsf{B} \times N_\mathsf{B}$ unitary matrices, while Δ is a $N_\mathsf{A} \times N_\mathsf{B}$ matrix with non-negative real entries and, precisely, with the only non-vanishing entries along the principal diagonal. Setting $\mathsf{d} \equiv \min\{N_\mathsf{A}, N_\mathsf{B}\}$, one can choose the unitary matrices U, V in such a way that the diagonal entries $(\delta_1, \delta_2, \ldots \delta_\mathsf{d})$ of Δ — the 'singular values' of ψ — are arranged in non-increasing order: $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_\mathsf{d}$ (we will always follow this convention for the singular values). The SVD (5) allows to write the state vector $|\psi\rangle$ in the Schmidt canonical form:

$$|\psi\rangle = \sum_{k=1}^{\mathsf{r}} \delta_k\, |\phi_k^\mathsf{A}\rangle|\phi_k^\mathsf{B}\rangle, \quad \mathsf{r} := \max\{k \in \{1,\ldots,\mathsf{d}\} :\ \delta_k > 0\}\,, \tag{6}$$

where r is the *Schmidt rank* of the vector $|\psi\rangle$ (observe that $\mathsf{r} = \mathrm{rank}(\psi)$), and the sets of vectors $\{|\phi_k^{\mathsf{A}}\rangle\}_{k=1}^{\mathsf{r}}$ and $\{|\phi_k^{\mathsf{B}}\rangle\}_{k=1}^{\mathsf{r}}$ are orthonormal systems, respectively in $\mathcal{H}_{\mathsf{A}}$ and $\mathcal{H}_{\mathsf{B}}$, determined by the unitary matrices U and V; indeed, we have that

$$|\psi\rangle = \sum_{n,\nu} \psi_{n\nu} |n\rangle|\nu\rangle = \sum_{n,\nu}\sum_{m,\mu} U_{nm}\,\Delta_{m\mu} V_{\mu\nu}\,|n\rangle|\nu\rangle, \tag{7}$$

hence:

$$|\phi_k^{\mathsf{A}}\rangle = \sum_{n=1}^{N_{\mathsf{A}}} U_{nk}\,|n\rangle, \quad |\phi_k^{\mathsf{B}}\rangle = \sum_{\nu=1}^{N_{\mathsf{B}}} V_{k\nu}\,|\nu\rangle, \quad k = 1, \ldots, \mathsf{r}. \tag{8}$$

Notice that the positive real Schmidt coefficients $\{\delta_k\}_{k=1}^{\mathsf{r}}$ are only constrained by the normalization of the state vector $|\psi\rangle$ to fulfill the condition: $\sum_k \delta_k^2 = 1$. We stress also that, although the matrix ψ depends on the choice of the orthonormal bases $\{|n\rangle\}_{n=1,\ldots,N_{\mathsf{A}}}$ and $\{|\nu\rangle\}_{\nu=1,\ldots,N_{\mathsf{B}}}$, its singular values — i.e. the Schmidt coefficients of $|\psi\rangle$ — are basis-independent.

The separability of pure states is related with the Schmidt coefficients: separable states have only a single non-vanishing Schmidt coefficient, i.e. $(\delta_1, \delta_2, \ldots, \delta_{\mathsf{d}}) = (1, 0, \ldots, 0)$ (in this case, $\mathsf{r} = \mathrm{rank}(\psi) = 1$); on the other hand, maximally entangled pure states have Schmidt coefficients $(\delta_1, \delta_2, \ldots, \delta_{\mathsf{d}}) = (1/\sqrt{\mathsf{d}}, 1/\sqrt{\mathsf{d}}, \ldots 1/\sqrt{\mathsf{d}})$ (in this case, $\mathsf{r} = \mathsf{d}$). In the general case, one can consider the Schmidt rank $\mathsf{r} = \mathrm{rank}(\psi)$ as an entanglement estimator (see refs. [21, 22, 23] for extensions of this approach to density operators).

In order to highlight the link between the Schmidt decomposition and the RC, it is convenient to describe the pure state $|\psi\rangle$ as a density operator (precisely, as a rank-one projector):

$$\hat{\rho}_\psi = |\psi\rangle\langle\psi| = \sum_{m,\mu,n,\nu} \psi_{m\mu}\psi^*_{n\nu}\,|m\rangle|\mu\rangle\langle n|\langle\nu|; \tag{9}$$

hence, with respect to the fixed orthonormal basis $\{|n\rangle|\nu\rangle\}$ in $\mathcal{H}$, the pure state $\hat{\rho}_\psi$ is now identified by the $(N_{\mathsf{A}}N_{\mathsf{B}}) \times (N_{\mathsf{A}}N_{\mathsf{B}})$ square matrix

$$\rho_\psi = \left[\rho_{\psi\,(m\mu)(n\nu)}\right], \quad \rho_{\psi\,(m\mu)(n\nu)} = \langle m|\langle\mu|\,\hat{\rho}_\psi|n\rangle|\nu\rangle = \psi_{m\mu}\psi^*_{n\nu}, \tag{10}$$

rather than by the $N_{\mathsf{A}} \times N_{\mathsf{B}}$ matrix ψ. Here the indexes $(m\mu)$ and $(n\nu)$ have to be regarded as *double indexes*; explicitly: $(m\mu) \leftrightarrow N_{\mathsf{B}}(m-1)+\mu$ and $(n\nu) \leftrightarrow N_{\mathsf{B}}(n-1)+\nu$.

With the square matrix ρ_ψ one can associate a *realigned* matrix ρ_ψ^{R}, which is a $N_{\mathsf{A}}^2 \times N_{\mathsf{B}}^2$ rectangular matrix with entries:

$$\rho^{\mathrm{R}}_{\psi\,(mn)(\mu\nu)} := \psi_{m\mu}\psi^*_{n\nu} = \rho_{\psi\,(m\mu)(n\nu)}, \tag{11}$$

where the double indexes $(mn) \leftrightarrow N_{\mathsf{A}}(m-1)+n$ and $(\mu\nu) \leftrightarrow N_{\mathsf{B}}(\mu-1)+\nu$ refer respectively to rows and columns of the matrix ρ_ψ^{R}. Thus, with the density matrix ρ_ψ, is associated the realigned matrix ρ_ψ^{R} having precisely the same entries, but *arranged in a different way*. It is immediate to check that — denoting by the symbol $\odot$ the *Kronecker product* of matrices (in order to avoid confusion with the tensor product $\otimes$ of the subsystems A and B) and by M^* the

complex conjugate of a matrix M (rather than the adjoint, which will be denoted by $M^{\dagger}$) — the following relation holds

$$\rho_{\psi}^{\mathrm{R}} = \psi \odot \psi^{*}. \tag{12}$$

Hence, considering the SVD (5) of the matrix ψ, and using the mixed product property of the Kronecker product, we obtain the following SVD of the realigned matrix:

$$\rho_{\psi}^{\mathrm{R}} = \mathcal{U}\,\Lambda\,\mathcal{V}, \quad \text{with: } \mathcal{U} = U \odot U^{*},\ \Lambda = \Delta \odot \Delta,\ \mathcal{V} = V \odot V^{*}, \tag{13}$$

where we have taken into account that $\Delta = \Delta^{*}$. Therefore, if the Schmidt coefficients of the state vector $|\psi\rangle$ are $\{\delta_k\}_{k=1}^{\mathsf{d}}$, then the associated realigned matrix ρ_{ψ}^{R} has singular values coinciding with the principal diagonal entries of the matrix $\Delta \odot \Delta$; namely: $\{\lambda_{(hk)} = \delta_h \delta_k\}$, where $(hk) \leftrightarrow \mathsf{d}(h-1)+k$. Notice that the rank of the realigned matrix ρ_{ψ}^{R} is given by: $\mathsf{R} = \mathrm{rank}(\Delta \odot \Delta) = \mathrm{rank}(\Delta)^2 = \mathrm{rank}(\psi)^2 \equiv \mathsf{r}^2$. Moreover, denoted by $\|\cdot\|_{\mathrm{tr}}$ the trace norm, we have that

$$\begin{aligned} \|\rho_{\psi}^{\mathrm{R}}\|_{\mathrm{tr}} := \mathrm{tr}(|\rho_{\psi}^{\mathrm{R}}|) &= \mathrm{tr}\Big(((\rho_{\psi}^{\mathrm{R}})^{\dagger}\rho_{\psi}^{\mathrm{R}})^{\frac{1}{2}}\Big) \\ &= \sum_{h,k=1}^{\mathsf{d}} \lambda_{(hk)} = \sum_{h,k=1}^{\mathsf{d}} \delta_h \delta_k = 1 + \sum_{h\neq k} \delta_h \delta_k\,; \end{aligned} \tag{14}$$

hence:

Proposition 1 *Let $\hat{\rho}_{\psi} = |\psi\rangle\langle\psi|$ be a pure state in $\mathcal{H} = \mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$. Then, the associated realigned matrix $\rho_{\psi}^{\mathrm{R}} = \psi \odot \psi^{*}$ satisfies:*

$$\|\rho_{\psi}^{\mathrm{R}}\|_{\mathrm{tr}} = \sum_{h,k=1}^{\mathsf{d}} \lambda_{(hk)} = 1 + \sum_{h\neq k} \delta_h \delta_k \geq 1. \tag{15}$$

Thus, inequality (15) is saturated if and only if the pure state $\hat{\rho}_{\psi}$ is separable:

$$\|\rho_{\psi}^{\mathrm{R}}\|_{\mathrm{tr}} = 1 \ \Leftrightarrow\ |\psi\rangle = |\psi^{\mathsf{A}}\rangle \otimes |\psi^{\mathsf{B}}\rangle. \tag{16}$$

For example, in the case of a two-qubit system, the density matrix ρ_{ψ} associated with the state vector (3) has the following form:

$$\rho_{\psi} = \begin{bmatrix} \psi_{11}\psi_{11}^{*} & \psi_{11}\psi_{12}^{*} & \psi_{11}\psi_{21}^{*} & \psi_{11}\psi_{22}^{*} \\ \psi_{12}\psi_{11}^{*} & \psi_{12}\psi_{12}^{*} & \psi_{12}\psi_{21}^{*} & \psi_{12}\psi_{22}^{*} \\ \psi_{21}\psi_{11}^{*} & \psi_{21}\psi_{12}^{*} & \psi_{21}\psi_{21}^{*} & \psi_{21}\psi_{22}^{*} \\ \psi_{22}\psi_{11}^{*} & \psi_{22}\psi_{12}^{*} & \psi_{22}\psi_{21}^{*} & \psi_{22}\psi_{22}^{*} \end{bmatrix}. \tag{17}$$

The corresponding realigned matrix is

$$\begin{aligned} \rho_{\psi}^{\mathrm{R}} &= \begin{bmatrix} \psi_{11}\psi_{11}^{*} & \psi_{11}\psi_{12}^{*} & \psi_{12}\psi_{11}^{*} & \psi_{12}\psi_{12}^{*} \\ \psi_{11}\psi_{21}^{*} & \psi_{11}\psi_{22}^{*} & \psi_{12}\psi_{21}^{*} & \psi_{12}\psi_{22}^{*} \\ \psi_{21}\psi_{11}^{*} & \psi_{21}\psi_{12}^{*} & \psi_{22}\psi_{11}^{*} & \psi_{22}\psi_{12}^{*} \\ \psi_{21}\psi_{21}^{*} & \psi_{21}\psi_{22}^{*} & \psi_{22}\psi_{21}^{*} & \psi_{22}\psi_{22}^{*} \end{bmatrix} \\ &= \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} \odot \begin{bmatrix} \psi_{11}^{*} & \psi_{12}^{*} \\ \psi_{21}^{*} & \psi_{22}^{*} \end{bmatrix} = \psi \odot \psi^{*}. \end{aligned} \tag{18}$$

Hence, if the singular values of the matrix ψ are (δ_1, δ_2), the realigned matrix (18) has singular values $(\delta_1^2 \geq \delta_1\delta_2 = \delta_2\delta_1 \geq \delta_2^2)$. If the Schmidt rank of the state vector $|\psi\rangle$ is r, then the rank of the realigned matrix ρ_ψ^{R} is $\mathsf{R} = \mathsf{r}^2 \in \{1, 4\}$. If $\hat{\rho}_\psi$ is a *separable* pure state, then ρ_ψ^{R} has only one non-vanishing singular value, i.e. $\{\lambda_{(hk)}\} = (1, 0, 0, 0)$ and $\mathsf{R} = 1$; moreover: $\|\rho_\psi^{\mathrm{R}}\|_{\mathrm{tr}} = 1$. If, otherwise, $\hat{\rho}_\psi$ is entangled, then $\mathsf{R} = 4$ and $\|\rho_\psi^{\mathrm{R}}\|_{\mathrm{tr}} > 1$; in particular, for a maximally entangled pure state we have: $\{\lambda_{(hk)}\} = (1/2, 1/2, 1/2, 1/2)$ and $\|\rho_\psi^{\mathrm{R}}\|_{\mathrm{tr}} = 2$.

Having established a relation between the separability of pure states and the singular values of the associated realigned matrices, we now proceed to extend this relation to a *generic* density operator in $\mathcal{H} = \mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$.

The first step is to observe that the association of a realigned matrix with a density operator is a straightforward generalization of the association introduced for pure states. Let $\hat{\rho}$ be a density operator in $\mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$ ($\hat{\rho} \geq 0$, $\mathrm{tr}(\hat{\rho}) = 1$), and let ρ denote the corresponding density matrix with respect to the fixed product basis $\{|n\rangle|\nu\rangle\}$ in $\mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$:

$$\rho = \left[\rho_{(m\mu)(n\nu)} = \langle m|\langle\mu|\,\hat{\rho}\,|n\rangle|\nu\rangle\right], \tag{19}$$

where $(m\mu) \leftrightarrow N_{\mathsf{B}}(m-1)+\mu$ and $(n\nu) \leftrightarrow N_{\mathsf{B}}(n-1)+\nu$. Then, as above, one can associate a $N_{\mathsf{A}}^2 \times N_{\mathsf{B}}^2$ realigned matrix ρ^{R} with the $N_{\mathsf{A}}N_{\mathsf{B}} \times N_{\mathsf{A}}N_{\mathsf{B}}$ density matrix ρ in the following way:

$$\rho^{\mathrm{R}}_{(mn)(\mu\nu)} := \rho_{(m\mu)(n\nu)}. \tag{20}$$

It is clear that the association $\hat{A} \mapsto A^{\mathrm{R}}$ (relative to the fixed product basis in $\mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$) is actually well defined for *any* linear operator $\hat{A}$ in $\mathcal{H} = \mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$.

For example, in the case of a two-qubit system, with a density matrix

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{bmatrix} \tag{21}$$

one can associate the realigned matrix

$$\rho^{\mathrm{R}} = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{21} & \rho_{22} \\ \rho_{13} & \rho_{14} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{41} & \rho_{42} \\ \rho_{33} & \rho_{34} & \rho_{43} & \rho_{44} \end{bmatrix}. \tag{22}$$

The next step is to observe that there is a precise link between the singular values of the realigned matrix ρ^{R} and the Schmidt coefficients of the density operator $\hat{\rho}$. Indeed, let us denote by $\hat{\mathcal{H}}$, $\hat{\mathcal{H}}_{\mathsf{A}}$ and $\hat{\mathcal{H}}_{\mathsf{B}}$ the Hilbert spaces of linear operators in $\mathcal{H}$, $\mathcal{H}_{\mathsf{A}}$ and $\mathcal{H}_{\mathsf{B}}$, respectively, endowed with the Hilbert-Schmidt (HS) scalar product (we are dealing with finite-dimensional vector spaces). Then, we have that

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\mathsf{A}} \otimes \hat{\mathcal{H}}_{\mathsf{B}}, \tag{23}$$

and one can consider the Schmidt decomposition of a density operator $\hat{\rho} \in \hat{\mathcal{H}}$ with respect to the tensor product decomposition (23). To this aim, let us fix a product (orthonormal) basis

in the Hilbert space $\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\mathsf{A}} \otimes \hat{\mathcal{H}}_{\mathsf{B}}$. It will be convenient to choose the basis formed by the partial isometries:

$$\hat{\mathrm{e}}_{(mn)(\mu\nu)} \equiv (|m\rangle\langle n|) \otimes (|\mu\rangle\langle\nu|), \quad |m\rangle\langle n| \in \hat{\mathcal{H}}_{\mathsf{A}},\ |\mu\rangle\langle\nu| \in \hat{\mathcal{H}}_{\mathsf{B}}, \tag{24}$$

with $m, n = 1, \ldots, N_{\mathsf{A}}$, $\mu, \nu = 1, \ldots, N_{\mathsf{B}}$. At this point, we can write:

$$\hat{\rho} = \sum_{m,n,\mu,\nu} c_{(mn)(\mu\nu)}\, \hat{\mathrm{e}}_{(mn)(\mu\nu)}, \tag{25}$$

where the coefficients of the expansion (25) are given by

$$\begin{aligned} c_{(mn)(\mu\nu)} = \left\langle \hat{\mathrm{e}}_{(mn)(\mu\nu)}, \hat{\rho} \right\rangle_{\hat{\mathcal{H}}} &= \mathrm{tr}\Big(\hat{\mathrm{e}}^{\dagger}_{(mn)(\mu\nu)}\, \hat{\rho}\Big) \\ &= \mathrm{tr}\Big(\hat{\mathrm{e}}_{(nm)(\nu\mu)}\, \hat{\rho}\Big) \\ &= \langle m|\langle\mu|\, \hat{\rho}\, |n\rangle|\nu\rangle \equiv \rho_{(m\mu)(n\nu)}. \end{aligned} \tag{26}$$

Hence, recalling relation (20), we conclude that the matrix of coefficients $\left[c_{(mn)(\mu\nu)}\right]$ coincides with the realigned matrix ρ^{R}. We have thus obtained the following:

Proposition 2 *For any density operator $\hat{\rho} \in \hat{\mathcal{H}}$, the Schmidt coefficients of $\hat{\rho}$, with respect to the tensor product decomposition $\hat{\mathcal{H}}_{\mathsf{A}} \otimes \hat{\mathcal{H}}_{\mathsf{B}}$ of $\hat{\mathcal{H}}$, coincide with the singular values of the associated realigned matrix ρ^{R}.*

It this clear that this result extends to all operators in $\mathcal{H}$: *for any linear operator $\hat{A}$ in $\mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$ — regarded as a vector of $\hat{\mathcal{H}}_{\mathsf{A}} \otimes \hat{\mathcal{H}}_{\mathsf{B}}$ — the Schmidt coefficients of $\hat{A}$ coincide with the singular values of the realigned matrix A^{R}*. This fact will be used in Sect. 3.

The last step is to extend the characterization of separable pure states in terms of the singular values of the associated realigned matrices to all states. Using Proposition 2, one easily gets to the RC. The RC provides a necessary condition for the separability of bipartite quantum states, which is known to be independent of the PPT criterion [16]. The RC relies on the evaluation of the trace norm of the realigned matrix ρ^{R}, which is equal to the sum of the singular values of ρ^{R}. We stress that, if $\hat{\rho}$ is a *mixed* (i.e. non-pure) state, then the SVD $\rho^{\mathrm{R}} = \mathcal{U}\,\Lambda\,\mathcal{V}$, with singular values $\{\lambda_k\}_{k=1,\ldots,\mathsf{d}^2}$, does not present the simple Kronecker product structure as in (13).

In order to obtain the RC, we argue as follows. Denoted by $\|\cdot\|_{\mathrm{tr}}$ the trace norm, we have:

$$\|\rho^{\mathrm{R}}\|_{\mathrm{tr}} := \mathrm{tr}(|\rho^{\mathrm{R}}|) = \mathrm{tr}\Big(((\rho^{\mathrm{R}})^{\dagger}\rho^{\mathrm{R}})^{\frac{1}{2}}\Big) = \sum_{k=1}^{\mathsf{d}^2} \lambda_k\,. \tag{27}$$

Now, in the special case of a simply separable state $\hat{\rho} = \hat{\rho}^{\mathsf{A}} \otimes \hat{\rho}^{\mathsf{B}}$, we have that the Schmidt coefficients of $\hat{\rho}$ (equivalently, the singular values of ρ^{R}) are given by:

$$(\lambda_1 = \|\hat{\rho}^{\mathsf{A}} \otimes \hat{\rho}^{\mathsf{B}}\|_{\hat{\mathcal{H}}}, \lambda_2 = 0, \ldots, \lambda_{\mathsf{d}^2} = 0), \tag{28}$$

where we denote by $\|\cdot\|_{\hat{\mathcal{H}}}$ (alternatively, by $\|\cdot\|_{\hat{\mathcal{H}}_{\mathsf{A}}}$ and $\|\cdot\|_{\hat{\mathcal{H}}_{\mathsf{B}}}$) the HS norm in $\hat{\mathcal{H}}$ (respectively, in $\hat{\mathcal{H}}_{\mathsf{A}}$ and $\hat{\mathcal{H}}_{\mathsf{B}}$). Observe that the following relation holds:

$$\|\hat{\rho}^{\mathsf{A}} \otimes \hat{\rho}^{\mathsf{B}}\|_{\hat{\mathcal{H}}} = \|\hat{\rho}^{\mathsf{A}}\|_{\hat{\mathcal{H}}_{\mathsf{A}}}\, \|\hat{\rho}^{\mathsf{B}}\|_{\hat{\mathcal{H}}_{\mathsf{B}}} = \sqrt{\mathrm{tr}((\hat{\rho}^{\mathsf{A}})^2)}\, \sqrt{\mathrm{tr}((\hat{\rho}^{\mathsf{B}})^2)}; \tag{29}$$

hence:

$$\|\hat{\rho}^{\mathsf{A}} \otimes \hat{\rho}^{\mathsf{B}}\|_{\hat{\mathcal{H}}} \leq 1, \text{ and } \|\hat{\rho}^{\mathsf{A}} \otimes \hat{\rho}^{\mathsf{B}}\|_{\hat{\mathcal{H}}} = 1 \Leftrightarrow \hat{\rho}^{\mathsf{A}}, \hat{\rho}^{\mathsf{B}} \text{ are pure states.} \tag{30}$$

Therefore, the Schmidt coefficients of $\hat{\rho}$ are characterized by the relations

$$\lambda_1 \leq 1,\ \lambda_2 = \ldots = \lambda_{\mathsf{d}^2} = 0, \text{ and } \lambda_1 = 1 \Leftrightarrow \hat{\rho}^{\mathsf{A}}, \hat{\rho}^{\mathsf{B}} \text{ are pure states.} \tag{31}$$

From these relations, recalling formula (27), we get the following:

Proposition 3 *If $\hat{\rho} \in \hat{\mathcal{H}}_{\mathsf{A}} \otimes \hat{\mathcal{H}}_{\mathsf{B}}$ is a simply separable state — $\hat{\rho} = \hat{\rho}^{\mathsf{A}} \otimes \hat{\rho}^{\mathsf{B}}$, $\hat{\rho}^{\mathsf{A}} \in \hat{\mathcal{H}}_{\mathsf{A}}$, $\hat{\rho}^{\mathsf{B}} \in \hat{\mathcal{H}}_{\mathsf{B}}$ — then the associated realigned matrix ρ^{R} satisfies:*

$$\|\rho^{\mathrm{R}}\|_{\mathrm{tr}} = \sum_{k=1}^{\mathsf{d}^2} \lambda_k = \lambda_1 \leq 1. \tag{32}$$

Inequality (32) is saturated if and only if the simply separable state $\hat{\rho}$ is pure:

$$\|\rho^{\mathrm{R}}\|_{\mathrm{tr}} = 1 \Leftrightarrow \hat{\rho} = \hat{\rho}^2. \tag{33}$$

Since separable states are convex superpositions of simply separable states, we can exploit Proposition 3 in order to obtain a separability criterion. To this aim, it will be convenient to introduce the following notations. Denoted by $\mathcal{M}(N_1, N_2)$ the vector space of $N_1 \times N_2$ complex-valued matrices, we define the *linear application*

$$\mathtt{RM} : \hat{\mathcal{H}} \to \mathcal{M}(N_{\mathsf{A}}^2, N_{\mathsf{B}}^2),\ \hat{A} \mapsto A^{\mathrm{R}}. \tag{34}$$

At this point, given a density operator $\hat{\rho} \in \hat{\mathcal{H}}$ — separable with respect to the tensor product decomposition $\mathcal{H} = \mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$, i.e.

$$\hat{\rho} = \sum_i p_i\ \hat{\rho}_i^{\mathsf{A}} \otimes \hat{\rho}_i^{\mathsf{B}}, \quad \hat{\rho}_i^{\mathsf{A}} \in \hat{\mathcal{H}}_{\mathsf{A}},\ \hat{\rho}_i^{\mathsf{B}} \in \hat{\mathcal{H}}_{\mathsf{B}}, \quad (\textit{finite} \text{ convex superposition}) \tag{35}$$

with $p_i > 0$ and $\sum_i p_i = 1$ — we can apply inequality (32) (for a simply separable state) and the triangle inequality:

$$\begin{aligned}
\|\mathtt{RM}(\hat{\rho})\|_{\mathrm{tr}} &= \|\mathtt{RM}(\textstyle\sum_i p_i\ \hat{\rho}_i^{\mathsf{A}} \otimes \hat{\rho}_i^{\mathsf{B}})\|_{\mathrm{tr}} \\
&= \|\textstyle\sum_i p_i\ \mathtt{RM}(\hat{\rho}_i^{\mathsf{A}} \otimes \hat{\rho}_i^{\mathsf{B}})\|_{\mathrm{tr}} && \text{(linearity of RM)} \\
&\leq \textstyle\sum_i p_i\ \|\mathtt{RM}(\hat{\rho}_i^{\mathsf{A}} \otimes \hat{\rho}_i^{\mathsf{B}})\|_{\mathrm{tr}} && \text{(triangle inequality)} \\
&\leq \textstyle\sum_i p_i = 1. && \text{(inequality (32))}
\end{aligned} \tag{36}$$

We have thus obtained the following result:

Theorem 1 ('realignment criterion') *Let $\hat{\rho}$ be a state in $\mathcal{H} = \mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$. If $\hat{\rho}$ is separable, then the associated realigned matrix ρ^{R} (relative to any product orthonormal basis in $\mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$) satisfies:*

$$\|\rho^{\mathrm{R}}\|_{\mathrm{tr}} = \sum_{k=1}^{\mathsf{d}^2} \lambda_k \leq 1. \tag{37}$$

Inequality (37) is saturated if the separable state $\hat{\rho}$ is pure.

3. A class of inequalities inducing new separability criteria

In the present section, we will derive a class of inequalities satisfied by all separable states of a bipartite quantum system. As in the case of the RC, these inequalities can be exploited for detecting entanglement. We will use arguments similar to the one adopted for deriving the standard RC, and we will keep the notations and the assumptions introduced in the preceding section. In particular, for any linear operator $\hat{A}$ in $\mathcal{H}$, we will denote by $\mathtt{RM}(\hat{A})$ the realigned matrix associated with $\hat{A}$, relatively to an arbitrarily fixed product orthonormal basis in $\mathcal{H}$ (recall also that $\mathtt{RM} : \hat{\mathcal{H}} \rightarrow \mathcal{M}(N_{\mathsf{A}}^2, N_{\mathsf{B}}^2)$ is a linear map). In view of the separability criteria introduced below, we stress also that the (easily computable) positive number $\|\mathtt{RM}(\hat{A})\|_{\mathrm{tr}}$ is equal to the trace norm of the matrix of coefficients of $\hat{A}$ — regarded as a vector of $\hat{\mathcal{H}}_{\mathsf{A}} \otimes \hat{\mathcal{H}}_{\mathsf{B}}$ — with respect to *any* product orthonormal basis in $\hat{\mathcal{H}}_{\mathsf{A}} \otimes \hat{\mathcal{H}}_{\mathsf{B}}$ (as we have seen, $\mathtt{RM}(\hat{A})$ is the matrix of coefficients associated with a basis of the special type (24)). Moreover, it is also worth mentioning the fact that the map

$$\|\mathtt{RM}(\cdot)\|_{\mathrm{tr}} : \hat{\mathcal{H}}_{\mathsf{A}} \otimes \hat{\mathcal{H}}_{\mathsf{B}} \rightarrow \mathbb{R}^+ \tag{38}$$

is a norm (actually, it is a *cross norm* on $\hat{\mathcal{H}}_{\mathsf{A}} \otimes \hat{\mathcal{H}}_{\mathsf{B}}$; see [17]). In the following, we will denote by $\mathcal{D}(\mathcal{H})$ ($\mathcal{D}(\mathcal{H}_{\mathsf{A}})$, $\mathcal{D}(\mathcal{H}_{\mathsf{B}})$) the convex subset of $\hat{\mathcal{H}}$ (respectively, of $\hat{\mathcal{H}}_{\mathsf{A}}$, $\hat{\mathcal{H}}_{\mathsf{B}}$) consisting of all density operators in $\mathcal{H}$ (respectively, in $\mathcal{H}_{\mathsf{A}}$, $\mathcal{H}_{\mathsf{B}}$).

Let $\hat{\rho} \in \mathcal{D}(\mathcal{H})$ be a separable state in the bipartite Hilbert space $\mathcal{H} = \mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$, with a separability decomposition of the form

$$\hat{\rho} = \sum_i p_i\, \hat{\rho}_i^{\mathsf{A}} \otimes \hat{\rho}_i^{\mathsf{B}}, \quad (\textit{finite}\ \text{convex superposition}) \tag{39}$$

where:

$$\sum_i p_i = 1, \ \text{ and } \ p_i > 0,\ \forall i. \tag{40}$$

Let us denote by $\hat{\rho}_{\mathsf{A}}$, $\hat{\rho}_{\mathsf{B}}$ the *marginals* (reduced density operators) of $\hat{\rho}$, namely:

$$\hat{\rho}_{\mathsf{A}} := \mathrm{tr}_{\mathsf{B}}(\hat{\rho}) = \sum_i p_i\, \hat{\rho}_i^{\mathsf{A}}, \quad \hat{\rho}_{\mathsf{B}} := \mathrm{tr}_{\mathsf{A}}(\hat{\rho}) = \sum_i p_i\, \hat{\rho}_i^{\mathsf{B}}. \tag{41}$$

Now, given $2\mathsf{n}$ linear or antilinear (super-)operators

$$\mathfrak{E}_1^{\mathsf{A}} : \hat{\mathcal{H}}_{\mathsf{A}} \rightarrow \hat{\mathcal{H}}_{\mathsf{A}},\ \ldots, \mathfrak{E}_{\mathsf{n}}^{\mathsf{A}} : \hat{\mathcal{H}}_{\mathsf{A}} \rightarrow \hat{\mathcal{H}}_{\mathsf{A}},\ \ \mathfrak{E}_1^{\mathsf{B}} : \hat{\mathcal{H}}_{\mathsf{B}} \rightarrow \hat{\mathcal{H}}_{\mathsf{B}},\ \ldots, \mathfrak{E}_{\mathsf{n}}^{\mathsf{B}} : \hat{\mathcal{H}}_{\mathsf{B}} \rightarrow \hat{\mathcal{H}}_{\mathsf{B}}, \tag{42}$$

with $\mathsf{n} \geq 1$ — precisely: we require these operators to be *jointly* linear or antilinear, i.e. either all linear or all antilinear, in such a way that one can consistently define tensor products and sums — we will associate with $\hat{\rho}$ the linear operator $\hat{\rho}(\mathfrak{E}_{1,\ldots,\mathsf{n}}^{\mathsf{A,B}}) : \mathcal{H} \rightarrow \mathcal{H}$ defined by

$$\begin{aligned}\hat{\rho}(\mathfrak{E}_{1,\ldots,\mathsf{n}}^{\mathsf{A,B}}) :=\ & \mathsf{n}^{-1} \left(\mathfrak{E}_1^{\mathsf{A}} \otimes \mathfrak{E}_1^{\mathsf{B}} + \cdots + \mathfrak{E}_{\mathsf{n}}^{\mathsf{A}} \otimes \mathfrak{E}_{\mathsf{n}}^{\mathsf{B}}\right)(\hat{\rho}) \\ & + \ \mathsf{n}^{-1}\Big(\sum_{k\neq l} \mathfrak{E}_k^{\mathsf{A}} \otimes \mathfrak{E}_l^{\mathsf{B}}\Big)(\hat{\rho}_{\mathsf{A}} \otimes \hat{\rho}_{\mathsf{B}}).\end{aligned} \tag{43}$$

Of course, $\hat{\rho}(\mathfrak{E}_{1,\ldots,\mathsf{n}}^{\mathsf{A,B}})$ will not be, in general, a density operator. Notice that the operator $\hat{\rho}(\mathfrak{E}_{1,\ldots,\mathsf{n}}^{\mathsf{A,B}})$ is defined in terms of the explicitly known operators $\hat{\rho}$, $\hat{\rho}_{\mathsf{A}}$ and $\hat{\rho}_{\mathsf{B}}$; i.e. definition (43) does not involve the (in general) ‘unknown density operators’ that appear in the r.h.s. of the

separability decomposition (39). It is easy to check, however, that the following relation holds:

$$\begin{aligned}\hat{\rho}(\mathfrak{E}^{\mathsf{A},\mathsf{B}}_{1,\dots,\mathsf{n}}) &= \frac{1}{\mathsf{n}} \sum_{i_1,\dots,i_\mathsf{n}} p_{i_1}\cdots p_{i_\mathsf{n}} \left(\mathfrak{E}^{\mathsf{A}}_1(\hat{\rho}^{\mathsf{A}}_{i_1}) + \cdots + \mathfrak{E}^{\mathsf{A}}_\mathsf{n}(\hat{\rho}^{\mathsf{A}}_{i_\mathsf{n}})\right) \otimes \left(\mathfrak{E}^{\mathsf{B}}_1(\hat{\rho}^{\mathsf{B}}_{i_1}) + \cdots + \mathfrak{E}^{\mathsf{B}}_\mathsf{n}(\hat{\rho}^{\mathsf{B}}_{i_\mathsf{n}})\right) \\ &\equiv \frac{1}{\mathsf{n}} \sum_{i_1,\dots,i_\mathsf{n}} p_{i_1}\cdots p_{i_\mathsf{n}} \left(\mathfrak{E}^{\mathsf{A}}_1(\hat{\rho}^{\mathsf{A}}_{i_1}) + \cdots + \mathfrak{E}^{\mathsf{A}}_\mathsf{n}(\hat{\rho}^{\mathsf{A}}_{i_\mathsf{n}})\right) \otimes (\mathsf{A}\rightarrow\mathsf{B}),\end{aligned} \tag{44}$$

where the symbol $(\mathsf{A}\rightarrow\mathsf{B})$ denotes repetition of the preceding term with the substitution of the subsystem A with the subsystem B (all the rest remaining unchanged). For instance, in order to check relation (44), consider that:

$$\begin{aligned}\sum_{i_1,\dots,i_\mathsf{n}} p_{i_1}\cdots p_{i_\mathsf{n}}\, \mathfrak{E}^{\mathsf{A}}_1(\hat{\rho}^{\mathsf{A}}_{i_1}) \otimes \mathfrak{E}^{\mathsf{B}}_1(\hat{\rho}^{\mathsf{B}}_{i_1}) &= \sum_{i_1} p_{i_1} \left(\mathfrak{E}^{\mathsf{A}}_1 \otimes \mathfrak{E}^{\mathsf{B}}_1\right)\left(\hat{\rho}^{\mathsf{A}}_{i_1} \otimes \hat{\rho}^{\mathsf{B}}_{i_1}\right) \\ &= \left(\mathfrak{E}^{\mathsf{A}}_1 \otimes \mathfrak{E}^{\mathsf{B}}_1\right)(\hat{\rho}),\end{aligned} \tag{45}$$

where, since $p_i \in \mathbb{R}$, both linearity and antilinearity work without distinction. In particular, for $\mathsf{n} = 1$, we have that $\hat{\rho}(\mathfrak{E}^{\mathsf{A},\mathsf{B}}_{1}) := \left(\mathfrak{E}^{\mathsf{A}}_1 \otimes \mathfrak{E}^{\mathsf{B}}_1\right)(\hat{\rho}) = \sum_i p_i \left(\mathfrak{E}^{\mathsf{A}}_1 \otimes \mathfrak{E}^{\mathsf{B}}_1\right)\left(\hat{\rho}^{\mathsf{A}}_i \otimes \hat{\rho}^{\mathsf{B}}_i\right)$, and, for $\mathsf{n} = 2$, we have:‡

$$\hat{\rho}(\mathfrak{E}^{\mathsf{A},\mathsf{B}}_{1,2}) := \frac{1}{2}\left(\mathfrak{E}^{\mathsf{A}}_1 \otimes \mathfrak{E}^{\mathsf{B}}_1 + \mathfrak{E}^{\mathsf{A}}_2 \otimes \mathfrak{E}^{\mathsf{B}}_2\right)(\hat{\rho}) + \frac{1}{2}\left(\mathfrak{E}^{\mathsf{A}}_1 \otimes \mathfrak{E}^{\mathsf{B}}_2 + \mathfrak{E}^{\mathsf{A}}_2 \otimes \mathfrak{E}^{\mathsf{B}}_1\right)(\hat{\rho}_\mathsf{A} \otimes \hat{\rho}_\mathsf{B}), \tag{46}$$

and

$$\begin{aligned}\hat{\rho}(\mathfrak{E}^{\mathsf{A},\mathsf{B}}_{1,2}) &= \frac{1}{2}\sum_{i,j} p_i p_j \left(\mathfrak{E}^{\mathsf{A}}_1(\hat{\rho}^{\mathsf{A}}_i) + \mathfrak{E}^{\mathsf{A}}_2(\hat{\rho}^{\mathsf{A}}_j)\right) \otimes \left(\mathfrak{E}^{\mathsf{B}}_1(\hat{\rho}^{\mathsf{B}}_i) + \mathfrak{E}^{\mathsf{B}}_2(\hat{\rho}^{\mathsf{B}}_j)\right) \\ &\equiv \frac{1}{2}\sum_{i,j} p_i p_j \left(\mathfrak{E}^{\mathsf{A}}_1(\hat{\rho}^{\mathsf{A}}_i) + \mathfrak{E}^{\mathsf{A}}_2(\hat{\rho}^{\mathsf{A}}_j)\right) \otimes (\mathsf{A}\rightarrow\mathsf{B}).\end{aligned} \tag{47}$$

At this point, in order to obtain the new class of inequalities announced above, we argue as follows. First of all, just for the sake of notational conciseness, we will consider in our derivation the operator $\hat{\rho}(\mathfrak{E}^{\mathsf{A},\mathsf{B}}_{1,2})$ rather than the more general expression $\hat{\rho}(\mathfrak{E}^{\mathsf{A},\mathsf{B}}_{1,\dots,\mathsf{n}})$ (the extension of our argument to $\hat{\rho}(\mathfrak{E}^{\mathsf{A},\mathsf{B}}_{1,\dots,\mathsf{n}})$ is straightforward). Then, we observe that

$$\begin{aligned}\|\mathtt{RM}(\hat{\rho}(\mathfrak{E}^{\mathsf{A},\mathsf{B}}_{1,2}))\|_{\mathrm{tr}} &= \frac{1}{2}\,\|\mathtt{RM}(\textstyle\sum_{i,j} p_i p_j \left(\mathfrak{E}^{\mathsf{A}}_1(\hat{\rho}^{\mathsf{A}}_i) + \mathfrak{E}^{\mathsf{A}}_2(\hat{\rho}^{\mathsf{A}}_j)\right) \otimes (\mathsf{A}\rightarrow\mathsf{B}))\|_{\mathrm{tr}} \\ &= \frac{1}{2}\,\|\textstyle\sum_{i,j} p_i p_j\, \mathtt{RM}((\mathfrak{E}^{\mathsf{A}}_1(\hat{\rho}^{\mathsf{A}}_i) + \mathfrak{E}^{\mathsf{A}}_2(\hat{\rho}^{\mathsf{A}}_j)) \otimes (\mathsf{A}\rightarrow\mathsf{B}))\|_{\mathrm{tr}} \\ &\le \frac{1}{2}\textstyle\sum_{i,j} p_i p_j\, \|\mathtt{RM}((\mathfrak{E}^{\mathsf{A}}_1(\hat{\rho}^{\mathsf{A}}_i) + \mathfrak{E}^{\mathsf{A}}_2(\hat{\rho}^{\mathsf{A}}_j)) \otimes (\mathsf{A}\rightarrow\mathsf{B}))\|_{\mathrm{tr}},\end{aligned} \tag{48}$$

where for obtaining the last line we have used the triangle inequality. Next, consider that the Schmidt coefficients of the operator $\left(\mathfrak{E}^{\mathsf{A}}_1(\hat{\rho}^{\mathsf{A}}_i) + \mathfrak{E}^{\mathsf{A}}_2(\hat{\rho}^{\mathsf{A}}_j)\right) \otimes (\mathsf{A}\rightarrow\mathsf{B}) \in \hat{\mathcal{H}}$ are given by

$$\left(\lambda_1 = \|(\mathfrak{E}^{\mathsf{A}}_1(\hat{\rho}^{\mathsf{A}}_i) + \mathfrak{E}^{\mathsf{A}}_2(\hat{\rho}^{\mathsf{A}}_j)) \otimes (\mathsf{A}\rightarrow\mathsf{B})\|_{\hat{\mathcal{H}}}, \lambda_2 = 0, \dots, \lambda_{\mathsf{d}^2} = 0\right), \tag{49}$$

‡ In the special case where the operators $\mathfrak{E}^{\mathsf{A}}_1, \mathfrak{E}^{\mathsf{B}}_1$ and $\mathfrak{E}^{\mathsf{A}}_2, \mathfrak{E}^{\mathsf{B}}_2$ coincide with the identity and -1 times the identity, respectively, the operator $\hat{\rho}(\mathfrak{E}^{\mathsf{A},\mathsf{B}}_{1,2})$ reduces to the expression $\hat{\rho} - \hat{\rho}_\mathsf{A} \otimes \hat{\rho}_\mathsf{B}$ which is central in the separability criterion obtained in ref. [18]; i.e.: $\|\mathtt{RM}(\hat{\rho} - \hat{\rho}_\mathsf{A} \otimes \hat{\rho}_\mathsf{B})\|_{\mathrm{tr}} \le \sqrt{(1 - \mathrm{tr}(\hat{\rho}^2_\mathsf{A}))\,(1 - \mathrm{tr}(\hat{\rho}^2_\mathsf{B}))}$, for every separable state $\hat{\rho} \in \mathcal{D}(\mathcal{H})$. See also Corollary 2 below.

since the Schmidt decomposition of this operator consists of a single term. Hence, we have:

$$\begin{aligned}\|\mathtt{RM}((\mathfrak{E}_1^{\mathsf{A}}(\hat{\rho}_i^{\mathsf{A}})+\mathfrak{E}_2^{\mathsf{A}}(\hat{\rho}_j^{\mathsf{A}}))\otimes(\mathsf{A}\to\mathsf{B}))\|_{\mathrm{tr}} &= \|\mathfrak{E}_1^{\mathsf{A}}(\hat{\rho}_i^{\mathsf{A}})+\mathfrak{E}_2^{\mathsf{A}}(\hat{\rho}_j^{\mathsf{A}})\|_{\hat{\mathcal{H}}_{\mathsf{A}}}\,\|\mathfrak{E}_1^{\mathsf{B}}(\hat{\rho}_i^{\mathsf{B}})+\mathfrak{E}_2^{\mathsf{B}}(\hat{\rho}_j^{\mathsf{B}})\|_{\hat{\mathcal{H}}_{\mathsf{B}}}\\ &= \Big(\langle\mathfrak{E}_1^{\mathsf{A}}(\hat{\rho}_i^{\mathsf{A}}),\mathfrak{E}_1^{\mathsf{A}}(\hat{\rho}_i^{\mathsf{A}})\rangle_{\hat{\mathcal{H}}_{\mathsf{A}}}+\langle\mathfrak{E}_2^{\mathsf{A}}(\hat{\rho}_j^{\mathsf{A}}),\mathfrak{E}_2^{\mathsf{A}}(\hat{\rho}_j^{\mathsf{A}})\rangle_{\hat{\mathcal{H}}_{\mathsf{A}}}\\ &\quad+(\langle\mathfrak{E}_1^{\mathsf{A}}(\hat{\rho}_i^{\mathsf{A}}),\mathfrak{E}_2^{\mathsf{A}}(\hat{\rho}_j^{\mathsf{A}})\rangle_{\hat{\mathcal{H}}_{\mathsf{A}}}+\mathrm{c.c.})\Big)^{\frac{1}{2}}(\mathsf{A}\to\mathsf{B})^{\frac{1}{2}},\end{aligned}\tag{50}$$

where $\langle\cdot,\cdot\rangle_{\hat{\mathcal{H}}_{\mathsf{A}}}$ is the (Hilbert-Schmidt) scalar product in $\hat{\mathcal{H}}_{\mathsf{A}}$. Thus, if we assume that, for some ϵ^{A}, $\epsilon^{\mathsf{B}}\geq 0$,

$$\|\mathfrak{E}_1^{\mathsf{A}}(\hat{\sigma}_1^{\mathsf{A}})\|^2_{\hat{\mathcal{H}}_{\mathsf{A}}}+\|\mathfrak{E}_2^{\mathsf{A}}(\hat{\sigma}_2^{\mathsf{A}})\|^2_{\hat{\mathcal{H}}_{\mathsf{A}}}\leq 2\,\epsilon^{\mathsf{A}},\quad \|\mathfrak{E}_1^{\mathsf{B}}(\hat{\sigma}_1^{\mathsf{B}})\|^2_{\hat{\mathcal{H}}_{\mathsf{B}}}+\|\mathfrak{E}_2^{\mathsf{B}}(\hat{\sigma}_2^{\mathsf{B}})\|^2_{\hat{\mathcal{H}}_{\mathsf{B}}}\leq 2\,\epsilon^{\mathsf{B}},\tag{51}$$

$\forall\,\hat{\sigma}_1^{\mathsf{A}},\hat{\sigma}_2^{\mathsf{A}}\in\mathcal{D}(\mathcal{H}_{\mathsf{A}})$, $\forall\,\hat{\sigma}_1^{\mathsf{B}},\hat{\sigma}_2^{\mathsf{B}}\in\mathcal{D}(\mathcal{H}_{\mathsf{B}})$, we find the following estimate:

$$\begin{aligned}\frac{\|\mathtt{RM}((\mathfrak{E}_1^{\mathsf{A}}(\hat{\rho}_i^{\mathsf{A}})+\mathfrak{E}_2^{\mathsf{A}}(\hat{\rho}_j^{\mathsf{A}}))\otimes(\mathsf{A}\to\mathsf{B}))\|_{\mathrm{tr}}}{2} &\leq \sqrt{\left(\epsilon^{\mathsf{A}}+\frac{1}{2}\Big(\langle\mathfrak{E}_1^{\mathsf{A}}(\hat{\rho}_i^{\mathsf{A}}),\mathfrak{E}_2^{\mathsf{A}}(\hat{\rho}_j^{\mathsf{A}})\rangle_{\hat{\mathcal{H}}_{\mathsf{A}}}+\mathrm{c.c.}\Big)\right)}\\ &\quad\times\sqrt{(\mathsf{A}\to\mathsf{B})}.\end{aligned}\tag{52}$$

Eventually, from inequalities (48) and (52), we get:

$$\begin{aligned}\|\mathtt{RM}(\hat{\rho}(\mathfrak{E}_{1,2}^{\mathsf{A},\mathsf{B}}))\|_{\mathrm{tr}} &\leq \textstyle\sum_{i,j}\sqrt{p_ip_j\left(\epsilon^{\mathsf{A}}+\frac{1}{2}\Big(\langle\mathfrak{E}_1^{\mathsf{A}}(\hat{\rho}_i^{\mathsf{A}}),\mathfrak{E}_2^{\mathsf{A}}(\hat{\rho}_j^{\mathsf{A}})\rangle_{\hat{\mathcal{H}}_{\mathsf{A}}}+\mathrm{c.c.}\Big)\right)}\\ &\quad\times\sqrt{p_ip_j\left(\epsilon^{\mathsf{B}}+\frac{1}{2}\Big(\langle\mathfrak{E}_1^{\mathsf{B}}(\hat{\rho}_i^{\mathsf{B}}),\mathfrak{E}_2^{\mathsf{B}}(\hat{\rho}_j^{\mathsf{B}})\rangle_{\hat{\mathcal{H}}_{\mathsf{B}}}+\mathrm{c.c.}\Big)\right)}\\ &\leq\sqrt{\left(\epsilon^{\mathsf{A}}+\frac{1}{2}\Big(\textstyle\sum_{i,j}p_ip_j\,\langle\mathfrak{E}_1^{\mathsf{A}}(\hat{\rho}_i^{\mathsf{A}}),\mathfrak{E}_2^{\mathsf{A}}(\hat{\rho}_j^{\mathsf{A}})\rangle_{\hat{\mathcal{H}}_{\mathsf{A}}}+\mathrm{c.c.}\Big)\right)}\\ &\quad\times\sqrt{(\mathsf{A}\to\mathsf{B})},\end{aligned}\tag{53}$$

where the second inequality above follows from the Cauchy-Schwarz inequality. Therefore, for every separable state $\hat{\rho}\in\mathcal{D}(\mathcal{H})$ we have:

$$\|\mathtt{RM}(\hat{\rho}(\mathfrak{E}_{1,2}^{\mathsf{A},\mathsf{B}}))\|_{\mathrm{tr}}\leq\sqrt{\left(\epsilon^{\mathsf{A}}+\frac{1}{2}\Big(\langle\mathfrak{E}_1^{\mathsf{A}}(\hat{\rho}_{\mathsf{A}}),\mathfrak{E}_2^{\mathsf{A}}(\hat{\rho}_{\mathsf{A}})\rangle_{\hat{\mathcal{H}}_{\mathsf{A}}}+\mathrm{c.c.}\Big)\right)}\,\sqrt{(\mathsf{A}\to\mathsf{B})}.\tag{54}$$

We notice that, if $\hat{\varrho}$ is a *generic* state in $\mathcal{D}(\mathcal{H})$ (i.e., separable or not), then one can define an operator $\hat{\varrho}(\mathfrak{E}_{1,2}^{\mathsf{A},\mathsf{B}})$ precisely as it has been done for the operator $\hat{\rho}(\mathfrak{E}_{1,2}^{\mathsf{A},\mathsf{B}})$ associated with a separable state $\hat{\rho}$. Moreover, observe that we have:

$$\begin{aligned}0\leq\frac{\|\mathtt{RM}((\mathfrak{E}_1^{\mathsf{A}}(\hat{\varrho}_{\mathsf{A}})+\mathfrak{E}_2^{\mathsf{A}}(\hat{\varrho}_{\mathsf{A}}))\otimes(\mathsf{A}\to\mathsf{B}))\|^2_{\mathrm{tr}}}{4} &\leq\left(\epsilon^{\mathsf{A}}+\frac{1}{2}\Big(\langle\mathfrak{E}_1^{\mathsf{A}}(\hat{\varrho}_{\mathsf{A}}),\mathfrak{E}_2^{\mathsf{A}}(\hat{\varrho}_{\mathsf{A}})\rangle_{\hat{\mathcal{H}}_{\mathsf{A}}}+\mathrm{c.c.}\Big)\right)\\ &\quad\times(\mathsf{A}\to\mathsf{B}),\end{aligned}\tag{55}$$

where $\hat{\varrho}_{\mathsf{A}}$, $\hat{\varrho}_{\mathsf{B}}$ are the marginals of $\hat{\varrho}$. Hence, it makes sense to check inequality (54) for a *generic* state (the square roots on the r.h.s. of inequality (54) are well defined). This inequality may not be satisfied by some state which can be then detected as an entangled state.

Extending the above proof to the operator $\hat{\rho}(\mathfrak{E}_{1,\ldots,\mathsf{n}}^{\mathsf{A},\mathsf{B}})$, $\mathsf{n}\geq 1$, associated with a separable state $\hat{\rho}\in\mathcal{D}(\mathcal{H})$, we obtain the following result:

Theorem 2 ('generalized RC') *Let $\hat{\rho}$ be a state in $\mathcal{H} = \mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$ and*

$$\mathfrak{E}_1^{\mathsf{A}} : \hat{\mathcal{H}}_{\mathsf{A}} \to \hat{\mathcal{H}}_{\mathsf{A}}, \ \ldots, \mathfrak{E}_{\mathsf{n}}^{\mathsf{A}} : \hat{\mathcal{H}}_{\mathsf{A}} \to \hat{\mathcal{H}}_{\mathsf{A}}, \ \ \mathfrak{E}_1^{\mathsf{B}} : \hat{\mathcal{H}}_{\mathsf{B}} \to \hat{\mathcal{H}}_{\mathsf{B}}, \ \ldots, \mathfrak{E}_{\mathsf{n}}^{\mathsf{B}} : \hat{\mathcal{H}}_{\mathsf{B}} \to \hat{\mathcal{H}}_{\mathsf{B}}, \quad (56)$$

$\mathsf{n} \geq 1$*, jointly linear or antilinear operators such that, for some* $\epsilon^{\mathsf{A}}, \epsilon^{\mathsf{B}} \geq 0$,

$$\|\mathfrak{E}_1^{\mathsf{A}}(\hat{\sigma}_1^{\mathsf{A}})\|_{\hat{\mathcal{H}}_{\mathsf{A}}}^2 + \cdots + \|\mathfrak{E}_{\mathsf{n}}^{\mathsf{A}}(\hat{\sigma}_{\mathsf{n}}^{\mathsf{A}})\|_{\hat{\mathcal{H}}_{\mathsf{A}}}^2 \leq \mathsf{n}\,\epsilon^{\mathsf{A}}, \quad \|\mathfrak{E}_1^{\mathsf{B}}(\hat{\sigma}_1^{\mathsf{B}})\|_{\hat{\mathcal{H}}_{\mathsf{B}}}^2 + \cdots + \|\mathfrak{E}_{\mathsf{n}}^{\mathsf{B}}(\hat{\sigma}_{\mathsf{n}}^{\mathsf{B}})\|_{\hat{\mathcal{H}}_{\mathsf{B}}}^2 \leq \mathsf{n}\,\epsilon^{\mathsf{B}}, \quad (57)$$

$\forall\, \hat{\sigma}_1^{\mathsf{A}}, \ldots, \hat{\sigma}_{\mathsf{n}}^{\mathsf{A}} \in \mathcal{D}(\mathcal{H}_{\mathsf{A}})$, $\forall\, \hat{\sigma}_1^{\mathsf{B}}, \ldots, \hat{\sigma}_{\mathsf{n}}^{\mathsf{B}} \in \mathcal{D}(\mathcal{H}_{\mathsf{B}})$*, and consider the linear operator* $\hat{\rho}(\mathfrak{E}_{1,\ldots,\mathsf{n}}^{\mathsf{A},\mathsf{B}})$ *in* $\mathcal{H}$ *defined by*

$$\hat{\rho}(\mathfrak{E}_{1,\ldots,\mathsf{n}}^{\mathsf{A},\mathsf{B}}) := \mathsf{n}^{-1}\Big(\sum_{k=1}^{\mathsf{n}} \mathfrak{E}_k^{\mathsf{A}} \otimes \mathfrak{E}_k^{\mathsf{B}}\,(\hat{\rho}) + \sum_{k\neq l} \mathfrak{E}_k^{\mathsf{A}} \otimes \mathfrak{E}_l^{\mathsf{B}}\,(\hat{\rho}_{\mathsf{A}} \otimes \hat{\rho}_{\mathsf{B}})\Big), \quad (58)$$

where $\hat{\rho}_{\mathsf{A}}$ *and* $\hat{\rho}_{\mathsf{B}}$ *are the marginals of* $\hat{\rho}$*, namely:* $\hat{\rho}_{\mathsf{A}} := \mathrm{tr}_{\mathsf{B}}(\hat{\rho})$, $\hat{\rho}_{\mathsf{B}} := \mathrm{tr}_{\mathsf{A}}(\hat{\rho})$*. If the state* $\hat{\rho}$ *is separable, then the following inequality holds:*

$$\|\mathtt{RM}(\hat{\rho}(\mathfrak{E}_{1,\ldots,\mathsf{n}}^{\mathsf{A},\mathsf{B}}))\|_{\mathrm{tr}} \leq \sqrt{\Big(\epsilon^{\mathsf{A}} + \frac{1}{\mathsf{n}}\sum_{k<l}\big(\langle\mathfrak{E}_k^{\mathsf{A}}(\hat{\rho}_{\mathsf{A}}), \mathfrak{E}_l^{\mathsf{A}}(\hat{\rho}_{\mathsf{A}})\rangle_{\hat{\mathcal{H}}_{\mathsf{A}}} + \mathrm{c.c.}\big)\Big)(\mathsf{A} \to \mathsf{B})}. \quad (59)$$

It is obvious that, once chosen the operators (56), inequality (59) is satisfied, in particular, if we set $\epsilon^{\mathsf{A}} = \breve{\epsilon}^{\mathsf{A}}$ and $\epsilon^{\mathsf{B}} = \breve{\epsilon}^{\mathsf{B}}$, where $\breve{\epsilon}^{\mathsf{A}}$ is the positive number defined by

$$\breve{\epsilon}^{\mathsf{A}} := \sup\Big\{\mathsf{n}^{-1}\sum_{k=1}^{\mathsf{n}} \|\mathfrak{E}_k^{\mathsf{A}}(\hat{\sigma}_k^{\mathsf{A}})\|_{\hat{\mathcal{H}}_{\mathsf{A}}}^2 \colon\ \hat{\sigma}_1^{\mathsf{A}}, \ldots, \hat{\sigma}_{\mathsf{n}}^{\mathsf{A}} \in \mathcal{D}(\mathcal{H}_{\mathsf{A}})\Big\}, \quad (60)$$

and $\breve{\epsilon}^{\mathsf{B}}$ is defined analogously.

The reason why we call Theorem 2 'generalized RC' is clear: for $\mathsf{n} = 1$, and $\mathfrak{E}_1^{\mathsf{A}}, \mathfrak{E}_1^{\mathsf{B}}$ coinciding with the identity (super-)operators (so that we can set: $\epsilon^{\mathsf{A}} = \epsilon^{\mathsf{B}} = 1$), we recover the RC; see also Corollary 2 below. We stress that, since it makes sense (as already observed) to check inequality (59) for a generic state, Theorem 2 induces, potentially, a whole new class of separability criteria. In this regard, it is easy to see that if one *fixes a priori* in the r.h.s. of each of inequalities (57) arbitrary strictly positive numbers ϵ^{A} and ϵ^{B} — e.g., if one sets $\epsilon^{\mathsf{A}} = \epsilon^{\mathsf{B}} = 1$ — then the class of *independent* separability criteria induced by inequality (59) is not restricted by such a specific choice. A certain criterion — i.e., a certain set of operators of the type (56) satisfying inequalities (57), and such that inequality (59) is violated by some entangled state — will be *optimal* if $\epsilon^{\mathsf{A}} = \breve{\epsilon}^{\mathsf{A}}$ and $\epsilon^{\mathsf{B}} = \breve{\epsilon}^{\mathsf{B}}$, with $\breve{\epsilon}^{\mathsf{A}}$, $\breve{\epsilon}^{\mathsf{B}}$ defined as in (60); otherwise, one can obtain an optimal criterion by suitably rescaling the operators (provided that one is able to evaluate the numbers $\breve{\epsilon}^{\mathsf{A}}$ and $\breve{\epsilon}^{\mathsf{B}}$).

Notice that, since the HS norm of density operators is not larger than one, condition (57) is satisfied if for the norm of the (super-)operators $\{\mathfrak{E}_k^{\mathsf{A}}, \mathfrak{E}_k^{\mathsf{B}}\}_{k=1,\ldots,\mathsf{n}}$ the following inequalities hold:

$$\sum_{k=1}^{\mathsf{n}} \|\mathfrak{E}_k^{\mathsf{A}}\|^2 \leq \mathsf{n}\,\epsilon^{\mathsf{A}}, \quad \sum_{k=1}^{\mathsf{n}} \|\mathfrak{E}_k^{\mathsf{B}}\|^2 \leq \mathsf{n}\,\epsilon^{\mathsf{B}}; \quad (61)$$

in particular, if the norm of the operators $\{\mathfrak{E}_k^{\mathsf{A}}, \mathfrak{E}_k^{\mathsf{B}}\}_{k=1,\ldots,\mathsf{n}}$ is not larger than $\sqrt{\epsilon^{\mathsf{A}}}$ or $\sqrt{\epsilon^{\mathsf{B}}}$, respectively. For instance, one can choose the operators $\{\mathfrak{E}_k^{\mathsf{A}}, \mathfrak{E}_k^{\mathsf{B}}\}_{k=1,\ldots,\mathsf{n}}$ to be either (all) unitary or (all) antiunitary in such a way that inequality (59) is satisfied with $\epsilon^{\mathsf{A}} = \epsilon^{\mathsf{B}} = 1$.

Observe that condition (57) is also satisfied — with $\epsilon^{\mathsf{A}} = \epsilon^{\mathsf{B}} = 1$ — if the (super-)operators $\{\mathfrak{E}_k^{\mathsf{A}}, \mathfrak{E}_k^{\mathsf{B}}\}_{k=1,\ldots,\mathsf{n}}$ are such that

$$\|\mathfrak{E}_k^{\mathsf{A}}(\hat{\sigma}^{\mathsf{A}})\|_{\hat{\mathcal{H}}_{\mathsf{A}}} \leq 1, \;\; \|\mathfrak{E}_k^{\mathsf{B}}(\hat{\sigma}^{\mathsf{B}})\|_{\hat{\mathcal{H}}_{\mathsf{B}}} \leq 1, \;\; k = 1, \ldots, \mathsf{n}, \tag{62}$$

$\forall\, \hat{\sigma}^{\mathsf{A}} \in \mathcal{D}(\mathcal{H}_{\mathsf{A}})$, $\forall\, \hat{\sigma}^{\mathsf{B}} \in \mathcal{D}(\mathcal{H}_{\mathsf{B}})$. One can assume, in particular, that they are trace-norm-nonincreasing on positive operators (since the Hilbert-Schmidt norm is majorized by the trace norm); for instance, positive trace-preserving linear maps.

Another natural choice consists in taking linear or antilinear operators $\{\hat{X}_k^{\mathsf{A}}, \hat{Y}_k^{\mathsf{A}}\}_{k=1,\ldots,\mathsf{n}}$ in $\mathcal{H}_{\mathsf{A}}$ and $\{\hat{X}_k^{\mathsf{B}}, \hat{Y}_k^{\mathsf{B}}\}_{k=1,\ldots,\mathsf{n}}$ in $\mathcal{H}_{\mathsf{B}}$, and setting:

$$\mathfrak{E}_k^{\mathsf{A}} : \hat{\mathcal{H}}_{\mathsf{A}} \ni \hat{A} \mapsto \hat{X}_k^{\mathsf{A}} \hat{A} \hat{Y}_k^{\mathsf{A}} \in \hat{\mathcal{H}}_{\mathsf{A}}, \quad \mathfrak{E}_k^{\mathsf{B}} : \hat{\mathcal{H}}_{\mathsf{B}} \ni \hat{B} \mapsto \hat{X}_k^{\mathsf{B}} \hat{B} \hat{Y}_k^{\mathsf{B}} \in \hat{\mathcal{H}}_{\mathsf{B}}, \quad k = 1, \ldots, \mathsf{n}\,; \tag{63}$$

precisely, we need to fix the following constraint: the operators $\{\hat{X}_k^{\mathsf{A}}, \hat{Y}_k^{\mathsf{A}}, \hat{X}_k^{\mathsf{B}}, \hat{Y}_k^{\mathsf{B}}\}_{k=1,\ldots,\mathsf{n}}$ must be linear or antilinear in such a way that the (super-)operators $\{\mathfrak{E}_k^{\mathsf{A}}, \mathfrak{E}_k^{\mathsf{B}}\}_{k=1,\ldots,\mathsf{n}}$ are (well defined as operators in $\hat{\mathcal{H}}_{\mathsf{A}}$, $\hat{\mathcal{H}}_{\mathsf{B}}$ and) jointly linear or antilinear. Hence, the operators $\{\hat{X}_k^{\mathsf{A}}, \hat{Y}_k^{\mathsf{A}}, \hat{X}_k^{\mathsf{B}}, \hat{Y}_k^{\mathsf{B}}\}_{k=1,\ldots,\mathsf{n}}$ must be jointly linear or antilinear as well: i.e., either all linear (so that the corresponding super-operators $\{\mathfrak{E}_k^{\mathsf{A}}, \mathfrak{E}_k^{\mathsf{B}}\}_{k=1,\ldots,\mathsf{n}}$ are linear) or all antilinear (in this case, the super-operators $\{\mathfrak{E}_k^{\mathsf{A}}, \mathfrak{E}_k^{\mathsf{B}}\}_{k=1,\ldots,\mathsf{n}}$ are antilinear).

Suppose, then, that the operators $\{\hat{X}_k^{\mathsf{A}}, \hat{Y}_k^{\mathsf{A}}, \hat{X}_k^{\mathsf{B}}, \hat{Y}_k^{\mathsf{B}}\}_{k=1,\ldots,\mathsf{n}}$ are jointly linear or antilinear. In this case, taking into account the fact that (due to a well known relation between the standard operator norm $\|\cdot\|$ and the HS norm)

$$\|\hat{X}_k^{\mathsf{A}} \hat{\sigma}^{\mathsf{A}} \hat{Y}_k^{\mathsf{A}}\|_{\hat{\mathcal{H}}_{\mathsf{A}}} \leq \|\hat{X}_k^{\mathsf{A}}\|\, \|\hat{Y}_k^{\mathsf{A}}\|\, \|\hat{\sigma}^{\mathsf{A}}\|_{\hat{\mathcal{H}}_{\mathsf{A}}}, \;\; \|\hat{X}_k^{\mathsf{B}} \hat{\sigma}^{\mathsf{B}} \hat{Y}_k^{\mathsf{B}}\|_{\hat{\mathcal{H}}_{\mathsf{B}}} \leq \|\hat{X}_k^{\mathsf{B}}\|\, \|\hat{Y}_k^{\mathsf{B}}\|\, \|\hat{\sigma}^{\mathsf{B}}\|_{\hat{\mathcal{H}}_{\mathsf{B}}}, \tag{64}$$

$k = 1, \ldots, \mathsf{n}$, for any $\hat{\sigma}^{\mathsf{A}} \in \mathcal{D}(\mathcal{H}_{\mathsf{A}})$ and $\hat{\sigma}^{\mathsf{B}} \in \mathcal{D}(\mathcal{H}_{\mathsf{B}})$ — where: $\|\hat{\sigma}^{\mathsf{A}}\|_{\hat{\mathcal{H}}_{\mathsf{A}}} \leq \|\hat{\sigma}^{\mathsf{A}}\|_{\mathrm{tr}} = 1$ and $\|\hat{\sigma}^{\mathsf{B}}\|_{\hat{\mathcal{H}}_{\mathsf{B}}} \leq 1$ — from Theorem 2 we obtain the following result:

Corollary 1 *Let $\hat{\rho}$ be a state in $\mathcal{H} = \mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$ and*

$$\hat{X}_1^{\mathsf{A}},\, \hat{Y}_1^{\mathsf{A}}, \ldots, \hat{X}_{\mathsf{n}}^{\mathsf{A}},\, \hat{Y}_{\mathsf{n}}^{\mathsf{A}} : \mathcal{H}_{\mathsf{A}} \to \mathcal{H}_{\mathsf{A}}, \quad \hat{X}_1^{\mathsf{B}},\, \hat{Y}_1^{\mathsf{B}}, \ldots, \hat{X}_{\mathsf{n}}^{\mathsf{B}},\, \hat{Y}_{\mathsf{n}}^{\mathsf{B}} : \mathcal{H}_{\mathsf{B}} \to \mathcal{H}_{\mathsf{B}}, \tag{65}$$

$\mathsf{n} \geq 1$, *jointly linear or antilinear operators such that*

$$\sum_{k=1}^{\mathsf{n}} \|\hat{X}_k^{\mathsf{A}}\|^2\, \|\hat{Y}_k^{\mathsf{A}}\|^2 \leq \mathsf{n}\, \epsilon^{\mathsf{A}}, \quad \sum_{k=1}^{\mathsf{n}} \|\hat{X}_k^{\mathsf{B}}\|^2\, \|\hat{Y}_k^{\mathsf{B}}\|^2 \leq \mathsf{n}\, \epsilon^{\mathsf{B}}, \tag{66}$$

for some $\epsilon^{\mathsf{A}}, \epsilon^{\mathsf{B}} > 0$, and consider the linear operator $\hat{\rho}(\hat{X}_{1,\ldots,\mathsf{n}}^{\mathsf{A},\mathsf{B}}, \hat{Y}_{1,\ldots,\mathsf{n}}^{\mathsf{A},\mathsf{B}})$ in $\mathcal{H}$ defined by

$$\begin{aligned}\hat{\rho}(\hat{X}_{1,\ldots,\mathsf{n}}^{\mathsf{A},\mathsf{B}}, \hat{Y}_{1,\ldots,\mathsf{n}}^{\mathsf{A},\mathsf{B}}) := {} & \frac{1}{\mathsf{n}} \sum_{k=1}^{\mathsf{n}} (\hat{X}_k^{\mathsf{A}} \otimes \hat{X}_k^{\mathsf{B}})\, \hat{\rho}\, (\hat{Y}_k^{\mathsf{A}} \otimes \hat{Y}_k^{\mathsf{B}}) \\ & + \frac{1}{\mathsf{n}} \sum_{k \neq l} (\hat{X}_k^{\mathsf{A}} \otimes \hat{X}_l^{\mathsf{B}})(\hat{\rho}_{\mathsf{A}} \otimes \hat{\rho}_{\mathsf{B}})(\hat{Y}_k^{\mathsf{A}} \otimes \hat{Y}_l^{\mathsf{B}}),\end{aligned} \tag{67}$$

where $\hat{\rho}_{\mathsf{A}}$ and $\hat{\rho}_{\mathsf{B}}$ are the marginals of $\hat{\rho}$. If $\hat{\rho}$ is separable, then the following inequality holds:

$$\begin{aligned}\|\mathtt{RM}(\hat{\rho}(\hat{X}_{1,\ldots,\mathsf{n}}^{\mathsf{A},\mathsf{B}}, \hat{Y}_{1,\ldots,\mathsf{n}}^{\mathsf{A},\mathsf{B}}))\|_{\mathrm{tr}} \leq {} & \sqrt{\Big(\epsilon^{\mathsf{A}} + \frac{1}{\mathsf{n}} \sum_{k<l} (\langle \hat{X}_k^{\mathsf{A}} \hat{\rho}_{\mathsf{A}} \hat{Y}_k^{\mathsf{A}}, \hat{X}_l^{\mathsf{A}} \hat{\rho}_{\mathsf{A}} \hat{Y}_l^{\mathsf{A}} \rangle_{\hat{\mathcal{H}}_{\mathsf{A}}} + \mathrm{c.c.})\Big)} \\ & \times \sqrt{\Big(\epsilon^{\mathsf{B}} + \frac{1}{\mathsf{n}} \sum_{k<l} (\langle \hat{X}_k^{\mathsf{B}} \hat{\rho}_{\mathsf{B}} \hat{Y}_k^{\mathsf{B}}, \hat{X}_l^{\mathsf{B}} \hat{\rho}_{\mathsf{B}} \hat{Y}_l^{\mathsf{B}} \rangle_{\hat{\mathcal{H}}_{\mathsf{B}}} + \mathrm{c.c.})\Big)}.\end{aligned} \tag{68}$$

In particular, we can set $\mathsf{n}=2$, and the operators $\{\hat{X}_k^{\mathsf{A}}, \hat{Y}_k^{\mathsf{A}}, \hat{X}_k^{\mathsf{B}}, \hat{Y}_k^{\mathsf{B}}\}_{k=1,2}$ can be chosen to be *unitary* (hence, we can set: $\epsilon^{\mathsf{A}} = \epsilon^{\mathsf{B}} = 1$). For the specific choice

$$\hat{X}_1^{\mathsf{A}} = e^{\mathrm{i}\alpha_1}\,\hat{\mathbb{I}}^{\mathsf{A}},\;\; \hat{X}_2^{\mathsf{A}} = e^{\mathrm{i}\alpha_2}\,\hat{\mathbb{I}}^{\mathsf{A}},\;\; \hat{X}_1^{\mathsf{B}} = e^{\mathrm{i}\beta_1}\,\hat{\mathbb{I}}^{\mathsf{B}},\;\; \hat{X}_2^{\mathsf{B}} = e^{\mathrm{i}\beta_2}\,\hat{\mathbb{I}}^{\mathsf{B}}, \tag{69}$$

$$\hat{Y}_1^{\mathsf{A}} = e^{\mathrm{i}\gamma_1}\,\hat{\mathbb{I}}^{\mathsf{A}},\;\; \hat{Y}_2^{\mathsf{A}} = e^{\mathrm{i}\gamma_2}\,\hat{\mathbb{I}}^{\mathsf{A}},\;\; \hat{Y}_1^{\mathsf{B}} = e^{\mathrm{i}\delta_1}\,\hat{\mathbb{I}}^{\mathsf{B}},\;\; \hat{Y}_2^{\mathsf{B}} = e^{\mathrm{i}\delta_2}\,\hat{\mathbb{I}}^{\mathsf{B}}, \tag{70}$$

— setting

$$\alpha_1 + \beta_1 + \gamma_1 + \delta_1 = \omega \tag{71}$$

and

$$-\alpha_1 + \alpha_2 - \gamma_1 + \gamma_2 = \theta, \quad -\beta_1 + \beta_2 - \delta_1 + \delta_2 = \phi, \tag{72}$$

so that we have:

$$\alpha_2 + \beta_2 + \gamma_2 + \delta_2 = \omega + \theta + \phi \tag{73}$$

and

$$\alpha_1 + \beta_2 + \gamma_1 + \delta_2 = \omega + \phi, \quad \alpha_2 + \beta_1 + \gamma_2 + \delta_1 = \omega + \theta \tag{74}$$

— we find the following result:

Corollary 2 *Let $\hat{\rho}$ be a state in $\mathcal{H} = \mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$. If $\hat{\rho}$ is separable, then the following inequality holds:*

$$\left\| \mathtt{RM}\left(\left(\frac{e^{\mathrm{i}\omega} + e^{\mathrm{i}(\omega+\theta+\phi)}}{2} \right) \hat{\rho} + \left(\frac{e^{\mathrm{i}(\omega+\theta)} + e^{\mathrm{i}(\omega+\phi)}}{2} \right) \hat{\rho}_{\mathsf{A}} \otimes \hat{\rho}_{\mathsf{B}} \right) \right\|_{\mathrm{tr}} \leq \sqrt{1 + \cos\theta \ \mathrm{tr}(\hat{\rho}_{\mathsf{A}}^2)} \times \sqrt{1 + \cos\phi \ \mathrm{tr}(\hat{\rho}_{\mathsf{B}}^2)}, \tag{75}$$

for any $\omega, \theta, \phi \in \mathbb{R}$, where $\hat{\rho}_{\mathsf{A}}$ and $\hat{\rho}_{\mathsf{B}}$ are the marginals of $\hat{\rho}$. In particular, for $\omega = 0$ and $\phi = -\theta$, we have that

$$\|\mathtt{RM}(\hat{\rho} + \cos\theta\, \hat{\rho}_{\mathsf{A}} \otimes \hat{\rho}_{\mathsf{B}})\|_{\mathrm{tr}} \leq \sqrt{(1 + \cos\theta \ \mathrm{tr}(\hat{\rho}_{\mathsf{A}}^2))\,(1 + \cos\theta \ \mathrm{tr}(\hat{\rho}_{\mathsf{B}}^2))}, \tag{76}$$

for any $\theta \in [0, \pi]$, while, for $\omega = 0$ and $\phi = \theta$, we have that

$$\left\| \mathtt{RM}\left(\left(\frac{1 + e^{\mathrm{i}2\theta}}{2} \right) \hat{\rho} + e^{\mathrm{i}\theta} \hat{\rho}_{\mathsf{A}} \otimes \hat{\rho}_{\mathsf{B}} \right) \right\|_{\mathrm{tr}} \leq \sqrt{(1 + \cos\theta \ \mathrm{tr}(\hat{\rho}_{\mathsf{A}}^2))\,(1 + \cos\theta \ \mathrm{tr}(\hat{\rho}_{\mathsf{B}}^2))}, \tag{77}$$

for any $\theta \in [0, 2\pi[$.

We remark that the presence of the parameter ω on the l.h.s. of inequality (75) is trivial. However, it is convenient for deriving specific inequalities; for instance, setting $\omega = -\theta = -\phi$, we get:

$$\|\mathtt{RM}(\cos\theta\, \hat{\rho} + \hat{\rho}_{\mathsf{A}} \otimes \hat{\rho}_{\mathsf{B}})\|_{\mathrm{tr}} \leq \sqrt{(1 + \cos\theta \ \mathrm{tr}(\hat{\rho}_{\mathsf{A}}^2))\,(1 + \cos\theta \ \mathrm{tr}(\hat{\rho}_{\mathsf{B}}^2))}, \quad \theta \in [0, \pi]. \tag{78}$$

Observe that, fixing the value $\theta = \pi/2$ in the family of inequalities (76), we find again the standard RC. For $\theta = \pi$, we recover the criterion derived in ref. [18], where it is also shown that this separability criterion is actually stronger than the standard RC. Notice that the same criterion can also be obtained, for instance, from the family of inequalities (77), or (78), setting again $\theta = \pi$.

4. Examples of entanglement detection

As already observed, Theorem 2 allows to obtain a new class of separability criteria for bipartite quantum systems. This class seems to be very large and a wide range exploration of the entanglement detection by means of such criteria is beyond the scope of the present contribution. However, for the sake of illustration, we will consider some simple example of application of our results.

In particular, the subclass of inequalities (76) induces a simple subclass of criteria characterized by the parameter $\theta \in [0, \pi]$. Varying this parameter, one obtains a 'continuous family of separability criteria', which includes, as already observed, the RC ($\theta = \pi/2$) and, for $\theta = \pi$, a criterion recently proposed in ref. [18]. Given a certain class of states, one can try to detect entanglement applying these criteria for different values of the parameter θ.

As an example, we have considered a one-parameter family of two-qutrit bound entangled states $\{\hat{\rho}(a)\}_{a\in[0,1]}$ presented in ref. [9]. In a given local basis for the two-qutrit system, this family of states have the following matrix representation:

$$\rho(a) = \frac{1}{8a+1}\begin{bmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{bmatrix}, \tag{79}$$

$0 \leq a \leq 1$. We have then considered a statistical mixture of this class of states with the maximally entangled state, hence obtaining the two-parameter family of states:

$$\hat{\rho}(a,p) = p\,\hat{\rho}(a) + \frac{1}{9}(1-p)\,\hat{\mathbb{I}}\,, \quad 0 \leq a \leq 1,\ 0 \leq p \leq 1\,. \tag{80}$$

We have checked numerically inequalities (76) on this set of states. As shown in figure 1, depending on the value of the parameter θ, the new separability criteria can be stronger than the RC. The numerical calculations also indicate that, on the family of states that we have considered, the strongest criterion for the detection of entanglement is the one corresponding to $\theta = \pi$.

One can consider another simple subclass of inequalities, namely, the class associated with the following set of super-operators:

$$\mathfrak{E}_1^{\mathsf{A}} = e^{\mathrm{i}\theta}\,\mathfrak{I}^{\mathsf{A}}, \ \ \mathfrak{E}_1^{\mathsf{B}} = e^{-\mathrm{i}\theta}\,\mathfrak{T}^{\mathsf{B}}, \ \ \mathfrak{E}_2^{\mathsf{A}} = \mathfrak{I}^{\mathsf{A}}, \ \ \mathfrak{E}_2^{\mathsf{B}} = \mathfrak{I}^{\mathsf{B}}, \ \ \theta \in [0,\pi], \tag{81}$$

where $\mathfrak{I}^{\mathsf{A}}\colon \hat{\mathcal{H}}_{\mathsf{A}} \rightarrow \hat{\mathcal{H}}_{\mathsf{A}}$, $\mathfrak{I}^{\mathsf{B}}\colon \hat{\mathcal{H}}_{\mathsf{B}} \rightarrow \hat{\mathcal{H}}_{\mathsf{B}}$ are the identity super-operators and $\mathfrak{T}^{\mathsf{B}}\colon \hat{\mathcal{H}}_{\mathsf{B}} \rightarrow \hat{\mathcal{H}}_{\mathsf{B}}$ is the transposition associated with a given orthonormal basis in $\mathcal{H}_{\mathsf{B}}$ (recall that transposition, differently from taking the adjoint, is a basis-dependent map), one obtains the following family of inequalities:

$$\|\mathtt{RM}(\hat{\rho}(\mathfrak{E}_{1,2}^{\mathsf{A},\mathsf{B}}))\|_{\mathrm{tr}} \leq \sqrt{(1+\cos\theta\ \mathrm{tr}(\hat{\rho}_{\mathsf{A}}^2))\,(1+\cos\theta\ \mathrm{tr}(\hat{\rho}_{\mathsf{B}}^{\mathsf{T}}\,\hat{\rho}_{\mathsf{B}}))}, \tag{82}$$

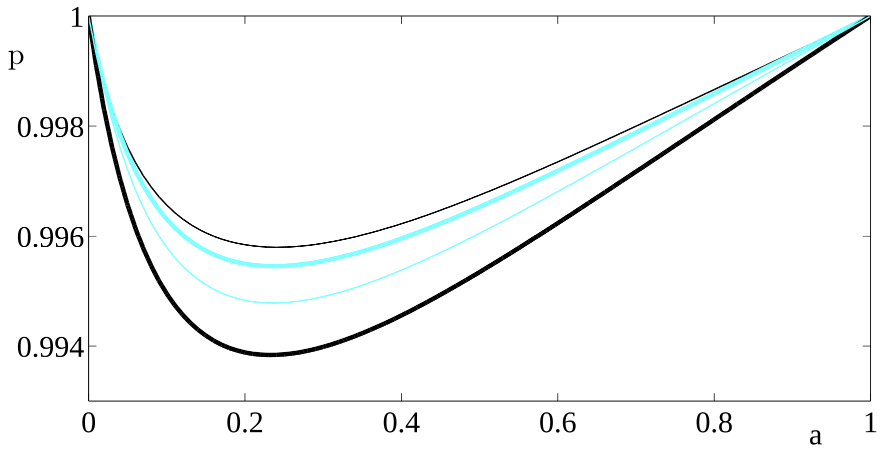

Figure 1. (color online.) The curves in the plot identify those elements of the family of states defined in (80) for which inequality (76) is saturated, in correspondence with various values of the parameter θ. The thin black line corresponds to $\theta = 0$, the thick light blue line to $\theta = \pi/2$ (RC), the thin light blue line to $\theta = 3\pi/4$, and the thick black line to $\theta = \pi$. The inequalities are violated by the (entangled) states associated with the values of the parameters a, p lying in the region above the curves. The criterion corresponding to $\theta = \pi$ turns out to be the strongest in detecting entangled states in the class of states considered.

with

$$\hat{\rho}(\mathfrak{E}^{\mathsf{A},\mathsf{B}}_{1,2}) = \frac{1}{2}\left(\hat{\rho}^{\mathsf{T}_\mathsf{B}} + \hat{\rho}\right) + \frac{1}{2}\left(e^{\mathrm{i}\theta}\hat{\rho}_\mathsf{A}\otimes\hat{\rho}_\mathsf{B} + e^{-\mathrm{i}\theta}\hat{\rho}_\mathsf{A}\otimes\hat{\rho}_\mathsf{B}^\mathsf{T}\right), \tag{83}$$

where $\hat{\rho}_\mathsf{B}^\mathsf{T}$ is the 'transposed operator' (i.e. $\hat{\rho}_\mathsf{B}^\mathsf{T} \equiv \mathfrak{T}^\mathsf{B}(\hat{\rho}_\mathsf{B})$), and $\hat{\rho}^{\mathsf{T}_\mathsf{B}}$ is the 'partially transposed operator' (i.e. $\hat{\rho}^{\mathsf{T}_\mathsf{B}} \equiv \mathfrak{I}^\mathsf{A}\otimes\mathfrak{T}^\mathsf{B}(\hat{\rho})$).

We have then considered the family of two-qubit states introduced in the second of papers [17]. In a given local basis for the two-qubit system, they are expressed by a matrix of the form:

$$\rho(t,s,r) = \frac{1}{2}\begin{bmatrix} 1+r & 0 & 0 & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & s-r & 0 \\ t & 0 & 0 & 1-s \end{bmatrix}. \tag{84}$$

For $r = s/2$, it is easy to show that these states are well defined in a domain containing the interval: $t \in [0, 0.25]$, $s \in [0, 0.9]$. Moreover, the specified family of states are known to be separable if and only if $t = 0$. As an example, we have checked inequalities (76) and (82) (in this case, the transposition $\mathfrak{T}^\mathsf{B}$ is the one associated with the given local basis), for different values of the parameter θ, on the specified family of states; namely, for $r = s/2$ with t, s belonging to the specified range: $t \in [0, 0.25]$, $s \in [0, 0.9]$. The corresponding plots are shown in figure 2.

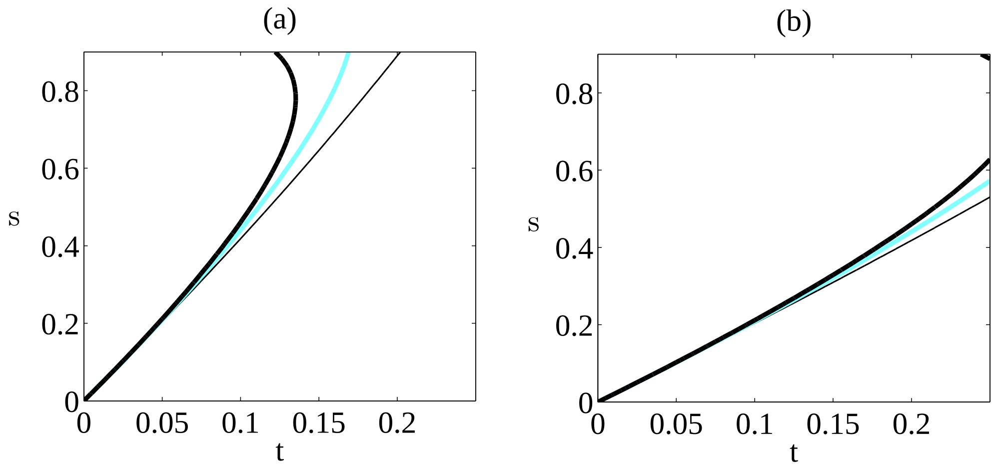

Figure 2. (color online.) The curves in the plot identify those elements of the family of states defined in (84) (with: $r = s/2$, $t \in [0, 0.25]$, $s \in [0, 0.9]$) for which inequality (76) (diagram (a)) and inequality (82) (diagram (b)) are saturated, in correspondence with three values of the parameter θ. The thin black line corresponds to $\theta = 0$, the thick light blue line to $\theta = 3\pi/4$, and the thick black line to $\theta = \pi$. The inequalities are violated by the (entangled) states associated with the values of the parameters t, s lying in the region on the right with respect to the curves.

5. Conclusions

In the present paper, we have introduced a class of inequalities for bipartite quantum systems that are satisfied by separable states and, hence, potentially induce new separability criteria. Each inequality corresponds to a choice of suitable linear or antilinear super-operators $\{\mathfrak{E}_k^{\mathsf{A}}\}_{k=1,\ldots,\mathsf{n}}$ and $\{\mathfrak{E}_k^{\mathsf{B}}\}_{k=1,\ldots,\mathsf{n}}$, respectively in the Hilbert-Schmidt spaces $\hat{\mathcal{H}}_{\mathsf{A}}$ and $\hat{\mathcal{H}}_{\mathsf{B}}$ associated with the 'local subsystems' A and B of the bipartite quantum system; see Theorem 2. A simple subclass of inequalities are parametrized, in a natural way, by $\theta \in [0, \pi]$; see inequality (76) in Corollary 2. This subclass contains, in particular, the inequality at the base of the standard RC ($\theta = \pi/2$), and an inequality ($\theta = \pi$) inducing a separability criterion which is the main result obtained in ref. [18], where it is shown that this criterion is actually stronger than the RC. We thus expect the class of separability criteria induced by the inequalities introduced here to be, in general, independent of the RC.

It is worth observing that another special subclass of inequalities is obtained setting $\mathsf{n} = 2$ and

$$\hat{X}_1^{\mathsf{A}} = \hat{F}^{\mathsf{A}}, \;\; \hat{Y}_1^{\mathsf{A}} = (\hat{F}^{\mathsf{A}})^\dagger, \;\; \hat{X}_1^{\mathsf{B}} = \hat{F}^{\mathsf{B}}, \;\; \hat{Y}_1^{\mathsf{B}} = (\hat{F}^{\mathsf{B}})^\dagger, \tag{85}$$

$$\hat{X}_2^{\mathsf{A}} = e^{\mathrm{i}\theta}\, \hat{F}^{\mathsf{A}}, \;\; \hat{Y}_2^{\mathsf{A}} = (\hat{F}^{\mathsf{A}})^\dagger, \;\; \hat{X}_2^{\mathsf{B}} = e^{-\mathrm{i}\theta}\, \hat{F}^{\mathsf{B}}, \;\; \hat{Y}_2^{\mathsf{B}} = (\hat{F}^{\mathsf{B}})^\dagger, \;\; \theta \in [0, \pi], \tag{86}$$

— where $\hat{F}^{\mathsf{A}} \colon \mathcal{H}_{\mathsf{A}} \rightarrow \mathcal{H}_{\mathsf{A}}$, $\hat{F}^{\mathsf{B}} \colon \mathcal{H}_{\mathsf{B}} \rightarrow \mathcal{H}_{\mathsf{B}}$ are linear operators such that $\|\hat{F}^{\mathsf{A}}\| \le 1$, $\|\hat{F}^{\mathsf{B}}\| \le 1$ (thus we can set $\epsilon^{\mathsf{A}} = \epsilon^{\mathsf{B}} = 1$) — in Corollary 1. Hence, for every separable state $\hat{\rho} \in \mathcal{D}(\mathcal{H})$, we have:

$$\left\| \mathtt{RM} \Big(\hat{F}^{\mathsf{A}} \otimes \hat{F}^{\mathsf{B}}\, \hat{\rho}\, (\hat{F}^{\mathsf{A}})^\dagger \otimes (\hat{F}^{\mathsf{B}})^\dagger + \cos\theta\, (\hat{F}^{\mathsf{A}} \hat{\rho}_{\mathsf{A}} (\hat{F}^{\mathsf{A}})^\dagger) \otimes (\hat{F}^{\mathsf{B}} \hat{\rho}_{\mathsf{B}} (\hat{F}^{\mathsf{B}})^\dagger) \Big) \right\|_{\mathrm{tr}} \le$$

$$\leq \sqrt{\Big(1+\cos\theta\ \mathrm{tr}\Big((\hat{F}^{\mathsf{A}}\hat{\rho}_{\mathsf{A}}(\hat{F}^{\mathsf{A}})^{\dagger})^2\Big)\Big)\ \Big(1+\cos\theta\ \mathrm{tr}\Big((\hat{F}^{\mathsf{B}}\hat{\rho}_{\mathsf{B}}(\hat{F}^{\mathsf{B}})^{\dagger})^2\Big)\Big)}. \tag{87}$$

For a suitable choice of the operators $\hat{F}^{\mathsf{A}}$, $\hat{F}^{\mathsf{B}}$ ('local filtering operations'), inequality (87) gives the 'local filtering enhancement' of the separability criterion induced by inequality (76). In particular, for $\theta = \pi/2$, one obtains the 'RC with local filtering'. In the case where $\dim(\mathcal{H}_{\mathsf{A}}) = \dim(\mathcal{H}_{\mathsf{B}})$, this powerful separability criterion is equivalent to the 'covariance matrix criterion under filtering'. See [24] and references therein. For $\theta = \pi$, we obtain the local filtering enhancement of the separability criterion introduced in ref. [18].

We stress that the new separability criteria induced by the inequalities introduced in the present paper are, in principle, practically implementable, since they involve easily computable quantities related to the density matrix and its marginals. Future work will be devoted to provide further examples and results along the lines traced in the present contribution.

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